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ON FREE VIBRATION ANALYSIS OF NON-LINEAR PIEZOELECTRIC CIRCULAR SHALLOW SPHERICAL SHELLS BY THE DIFFERENTIAL QUADRATURE ELEMENT METHOD

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1. INTRODUCTION

Piezoelectric materials are widely used as sensors and actuators in smart materials and structures due to their fast responses and some other advantages. As an actuator, it can be used to control structural shapes or to suppress undesired vibrations in certain degrees for some flexible lightweight structural elements or structures. In such cases, geometrical non-linearity should be considered in the theoretical formulations.

Recently, the axisymmetric free vibration analysis were performed by using the semi-analytical and semi-numerical method for simply supported non-linear piezoelectric circular plates [1] and simply supported circular shallow spherical shells [2]. Since analytical solutions can only be obtained for some simple cases, numerical methods should be adopted to obtain responses of smart structural elements or smart structures under various loading conditions.

The present authors recently proposed a new method called the differential quadrature element method (DQEM) and performed static, buckling and free vibration analyses of frame structures [3] and fundamental frequency of circular plates with stepped thickness by using DQEM [4]. It should be mentioned that recently Liew and his colleagues also independently proposed the differential quadrature element method (DQEM) [5–9] and performed static and free vibration analysis of Reissner–Mindlin rectangular and polar plates by using DQEM [5–9]. The ideas for both methods with the same name are similar, the main difference is that Wang *et al.* [3, 4] focused on the thin plates, while Liew *et al.* [5–9] focused on the moderately thick and thick plates. The numerical examples indicate that the DQEM is also very convenient to use and can yield very accurate results with small computational effort for plates with discontinuities on loading, geometry and boundary conditions. In view of the fact that the previous researchers have not analyzed the fundamental frequency of circular plates and circular shallow spherical shells with piezoelectric actuators by DQEM, the writers have computed frequencies using DQEM. The problem being considered in this paper involves plates and shallow shells of an isotropic material. The material and electrical properties of the actuator are also isotropic. Geometrical non-linearity but with small amplitudes are considered for the free vibration analysis. For completeness, the theoretical formulations for the shallow shells are briefly given, followed by the detailed solution procedures and formulations by DQEM. The

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circular plate is treated as a special case. The title problem is analyzed and the DQEM results are compared with the results in the open literature [1, 2] to show the applicability of the DQEM.

2. GOVERNING DIFFERENTIAL EQUATIONS

For simplicity and without loss of generality, only axisymmetric deformation of the piezoelectric circular shallow spherical shell is considered in this paper. The shell is composed of three layers, the upper and lower layers are uniform this piezoelectric materials. Both materials are isotropic in mechanical and/or electric properties. A von Karman-type geometric relation is adopted. The well-known governing differential equations are

$$N_r - N_\theta + r\frac{\partial N_r}{\partial r} = 0,$$

$$M_r + r\frac{\partial M_r}{\partial r} - M_\theta - rQ_r = 0, \tag{1}$$

$$\frac{\partial}{\partial r}(rQ_r) + \frac{\partial}{\partial r}\left[rN_r\left(\frac{\mathrm{d}z}{\mathrm{d}r} + \frac{\partial w}{\partial r}\right)\right] - \rho hr\frac{\partial^2 w}{\partial r^2} = 0,$$

where Q_r is the shear force in the r direction, ρ, h are the mass density and shell's thickness, z denotes the initial position of the shallow shell computed by

$$z(r) = -f\left(1 - \frac{r^2}{a^2}\right), \tag{2}$$

where f and a are the height of the shell and the radius of the circular edge. The total in-plane forces N_r, N_θ and the bending moments M_r, M_θ are computed by

$$N_r = N_r^m - N_r^e, \quad N_\theta = N_\theta^m - N_\theta^e,$$

$$M_r = M_r^m - M_r^e, \quad M_\theta = M_\theta^m - M_\theta^e, \tag{3}$$

in which superscripts m, e represent the mechanical and electrical quantities respectively.

After some manipulations, the non-linear governing differential equations can be expressed in terms of $N(= rN_r^m)$ and w as follows:

$$D\left(\frac{\partial^4 w}{\partial r^4} + \frac{2}{r}\frac{\partial^3 w}{\partial r^3} - \frac{1}{r^2}\frac{\partial^2 w}{\partial r^2} + \frac{1}{r^3}\frac{\partial w}{\partial r}\right)$$

$$= -\rho h\frac{\partial^2 w}{\partial t^2} + \frac{1}{r}\frac{\partial}{\partial r}\left[N\left(\frac{2fr}{a^2} + \frac{\partial w}{\partial r}\right)\right] - \frac{1}{r}\frac{\partial}{\partial r}\left[rN_r^e\left(\frac{2fr}{a^2} + \frac{\partial w}{\partial r}\right)\right] - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial M_r^e}{\partial r}\right),$$

$$r\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\right](rN) = -Eh\left[\frac{1}{2}\left(\frac{\partial w}{\partial r}\right)^2 + \frac{2fr}{a^2}\frac{\partial w}{\partial r}\right] + \frac{\partial}{\partial r}\left(r^2\frac{\partial N_r^e}{\partial r}\right) + \mu r\frac{\partial N_r^e}{\partial r}, \tag{4}$$

where $0 \leqslant r \leqslant a$, $D = Eh^3/12(1-\mu^2)$ is the bending stiffness, E, μ, w are Young's modulus, the Poisson ratio and deflection respectively.

The responses can be decomposed into two parts, one is the static part independent of the time and the other is the dynamic part depending on the time, namely,

$$w(r,t) = w_s(r) + w_t(r,t), \quad N(r,t) = N_s(r) + N_t(r,t). \tag{5}$$

Substituting equation (5) into equation (4) yields the governing differential equations (equation (6)) for static deformations, namely,

$$\begin{aligned} & D\left(\frac{\partial^4 w_s}{\partial r^4} + \frac{2}{r}\frac{\partial^3 w_s}{\partial r^3} - \frac{1}{r^2}\frac{\partial^2 w_s}{\partial r^2} + \frac{1}{r^3}\frac{\partial w_s}{\partial r}\right) \\ & \quad = +\frac{1}{r}\frac{\partial}{\partial r}\left[N_s\left(\frac{2fr}{a^2} + \frac{\partial w_s}{\partial r}\right)\right] - \frac{1}{r}\frac{\partial}{\partial r}\left[rN_r^e\left(\frac{2fr}{a^2} + \frac{\partial w_s}{\partial r}\right)\right] - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial M_r^e}{\partial r}\right), \\ & r\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\right](rN_s) = -Eh\left[\frac{1}{2}\left(\frac{\partial w_s}{\partial r}\right)^2 + \frac{2fr}{a^2}\frac{\partial w_s}{\partial r}\right] + \frac{\partial}{\partial r}\left(r^2\frac{\partial N_r^e}{\partial r}\right) + \mu r\frac{\partial N_r^e}{\partial r} \end{aligned} \tag{6}$$

and the governing differential equations (equation (7)) for dynamic responses, i.e.,

$$\begin{aligned} & D\left(\frac{\partial^4 w_t}{\partial r^4} + \frac{2}{r}\frac{\partial^3 w_t}{\partial r^3} - \frac{1}{r^2}\frac{\partial^2 w_t}{\partial r^2} + \frac{1}{r^3}\frac{\partial w_t}{\partial r}\right) \\ & \quad = -\rho h\frac{\partial^2 w_t}{\partial t^2} + \frac{1}{r}\frac{\partial}{\partial r}\left[N_t\left(\frac{2fr}{a^2} + \frac{\partial w_s}{\partial r}\right)\right] + \frac{1}{r}\frac{\partial}{\partial r}\left[N_s\frac{\partial w_t}{\partial r}\right] + \frac{1}{r}\frac{\partial}{\partial r}\left[N_t\frac{\partial w_t}{\partial r}\right], \\ & r\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\right](rN_t) = -Eh\left[\left(\frac{\partial w_s}{\partial r}\right)\left(\frac{\partial w_t}{\partial r}\right) + \frac{2fr}{a^2}\frac{\partial w_t}{\partial r} + \frac{1}{2}\left(\frac{\partial w_t}{\partial r}\right)^2\right], \end{aligned} \tag{7}$$

where the electric quantities are assumed to be independent of time for the free vibration analysis. For a small-amplitude free vibration in the vicinity of the non-linear static position, the non-linear terms in the dynamic equations, i.e., the last term in both equations of equation (7), can be dropped. The boundary conditions can be also decomposed into two parts. This is straightforward and details may be found in references [2, 10].

3. SOLUTION PROCEDURES BY DQEM

For demonstration and comparison purposes, one considers the cases of small-amplitude free vibrations of the shell in the vicinity of the non-linear static position. The shell is simply supported at its edge and uniformly distributed voltages but with opposite sign are applied in the upper and lower actuators. Thus $N_r^e = 0$, and M_r^e is a constant. For this case, w_t can be assumed as

$$w_t(r,t) = W(r)\sin\omega t, \tag{8}$$

where ω is the circular frequency.

In order to obtain the solutions for the shells, the non-linear static state has to be obtained first. This can be accomplished by using the DQEM together with the

Newton–Raphson method to solve equation (6). Due to space limitations, details can be found in reference [10] and are omitted here.

Let A_{ij}, B_{ij}, C_{ij} and D_{ij} be the weighting coefficients for the first, second, third, and fourth order derivatives of the unknown deflection w (both w_s and W), and A^*_{ij}, B^*_{ij} be the first and second order derivatives of the unknown in-plane force N (both N_s and N_t). Applying the DQEM to the dynamic governing differential equations (equation (7)) and regularity conditions at the shell center yields the following algebraic equations in terms of W^*_i and N_i, namely,

$$\sum_{j=1}^{NP} A_{1,j} W^*_j = 0,$$

$$D\left(\sum_{j=1}^{NP} D_{ij} + \frac{2}{r_i}\sum_{j=1}^{NP} C_{ij} - \frac{1}{r_i^2}\sum_{j=1}^{NP} B_{ij} + \frac{1}{r_i^3}\sum_{j=1}^{NP} A_{ij}\right) W^*_j - \left(\frac{2f}{a^2} + \frac{1}{r_i}\sum_{j=1}^{NP} A_{ij}\bar{w}^*_j\right)\sum_{j=1}^{NP1} A^*_{ij} N_j$$

$$-\frac{N_i}{r_i}\left(\frac{2f}{a^2} + \sum_{j=1}^{NP} B_{ij}\bar{w}^*_j\right) - \frac{1}{r_i}\left(\sum_{j=1}^{NP1} A^*_{ij}\bar{N}_j\right)\sum_{j=1}^{NP} A_{ij} W^*_j - \frac{\bar{N}_i}{r_i}\sum_{j=1}^{NP} B_{ij} W^*_j = \rho h \omega^2 W,$$

$$(i = 2, 3, \ldots, NP1 - 1),$$

$$-D\left(\sum_{j=1}^{NP} C_{NP1,j} + \frac{1}{a}\sum_{j=1}^{NP} B_{NP1,j} - \frac{1}{a^2}\sum_{j=1}^{NP} A_{NP1,j}\right) W^*_j = 0,$$

$$D\left(\sum_{j=1}^{NP} B_{NP1,j} + \frac{\mu}{a}\sum_{j=1}^{NP} A_{NP1,j}\right) W^*_j = 0,$$

$$N_1 = 0, \tag{9}$$

$$\left(r_i\sum_{j=1}^{NP1} B^*_{ij} + \sum_{j=1}^{NP1} A^*_{ij}\right) N_j - \frac{N_i}{r_i} + Eh\left(\left[\sum_{j=1}^{NP} A_{ij}\bar{w}^*_j\right]\left[\sum_{j=1}^{NP} A_{ij} W^*_j\right] + \frac{2fr_i}{a^2}\sum_{j=1}^{NP} A_{ij} W^*_j\right) = 0$$

$$(i = 2, 3, \ldots, NP1 - 1),$$

$$\sum_{j=1}^{NP1} A^*_{NP1,j} N_j - \frac{\mu}{a} N_{NP1} = 0,$$

where $NP1$ is the total number of grid points in the r direction and $NP = NP1 + 1$, r_i, W_j, $\bar{w}_i$, N_i, $\bar{N}_i$ are the values of r, W, w_s, N_t, N_s at the grid point i respectively. Note that W^*_i, $\bar{w}^*_i$ contains the values of W, w_s at all grid points $(W_i, \bar{w}_i)$ and its first derivatives at grid point $NP1$. The differential quadrature element method and detailed procedures to obtain various weighting coefficients are given by Wang *et al.* [3, 4, 10]. The explicit fomulae to compute these weighting coefficients are also provided in the open literature [11, 12]. It can be seen that there are $NP + NP1$ linear algebraic equations in equation (9). Equation (9) is a generalized eigen value problem, $KX = \lambda MX$, which can be solved by using standard solvers to obtain the eigenvalues and eigenvectors without any difficulties.

4. RESULTS AND DISCUSSIONS

A Fortran computer program is written, and the free vibration of non-linear piezoelectric spherical shallow shells is reanalyzed by the DQEM. Gauss–Lobatto formula is used to

TABLE 1

Variation of central deflection and frequencies with the number of grid points ($\bar{\omega}_1 = \omega_1 a^2\sqrt{\rho h/D}$; $\bar{\omega}_2 = \omega_{21} a^2\sqrt{\rho h/D}$; $\phi^* = 0{\cdot}4; f = 0$)

Grid number $NP1$	$W(0)/h$	$\bar{\omega}_1$	$\bar{\omega}_2$
10	0·72676068	9·6331612	33·251183
11	0·72676100	9·6191393	33·244581
12	0·72676104	9·6087966	33·240003
13	0·72676111	9·6009346	33·236687
14	0·72676072	9·5948231	33·234109
15	0·72676196	9·5899804	33·232079

determine the non-uniform grid points in the analysis, namely,

$$\frac{r_i}{a} = \frac{1}{2} - \frac{1}{2}\cos\frac{(i-1)\pi}{NP1-1} \quad (i = 1, 2, \ldots, NP1), \tag{10}$$

where $NP1$ is the number of grid points. Previous research shows that equation (10) can yield reliable and accurate results even for problems sensitive to grid spacing. A convergence study is performed to choose the right number of grid points to achieve the required accuracy. The numerical results with different grids for simply supported circular plates under control voltage $\phi^* = 4$ are listed in Table 1. For comparison, two parameters are introduced, namely,

$$\gamma = \sqrt{3(1-\mu^2)}\,\frac{f}{h}, \quad \phi^* = 6(1-\mu^2)\sqrt{3(1-\mu^2)}\,\frac{a^2 e_{31}(h+h_p)\phi_3^U}{Eh^4}, \tag{11}$$

where e_{31}, h_p, ϕ_3^U are the piezoelectric constants, thickness of the piezoelectric material and the control voltage applied on the upper surface of the shell respectively. It can be seen from Table 1 that the variations are within 0·1 per cent for the central deflections as well as the

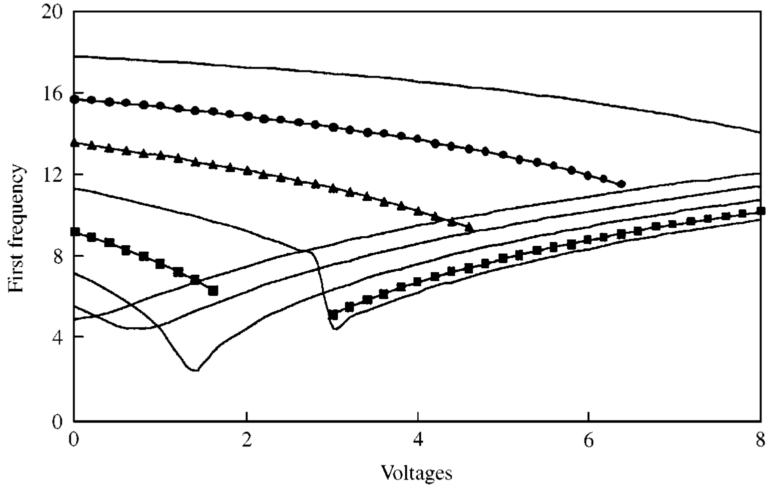

Figure 1. Variations of the first frequency ($\bar{\omega}_1 = \omega_1 a^2\sqrt{\rho h/D}$) with control voltages ϕ^* (equation (11)).

first and second frequencies with grid points greater than 12. It is also seen that the variations are much smaller for the deflections than for the frequencies. The grid number is set to 15 for the results presented in this paper.

The free vibration of shallow shells with different heights under control voltages is analyzed. Figure 1 shows the variation of the first frequency ($\bar{\omega}_1 = \omega_1 a^2\sqrt{\rho h/D}$) with control voltages for shells with different heights, including the circular plates ($\gamma = 0$). There are eight curves corresponding to $\gamma = 0$, 0·5, 1, 1·5, 2, 2·5, 3, 3·5 for curves from the lowest to the top at $\phi^* = 0$. Due to snapping, no convergent results are obtained for the non-linear static equations if the control voltages are gradually increased for certain cases. Thus, curves with symbols corresponding to $\gamma = 1{\cdot}5$, 2·5, 3 are not completed in the entire range in Figure 1. The DQEM results agree well with the semi-analytical and semi-numerical solutions in references [1, 2]. It should be mentioned, however, that the DQEM is easy to use, can also obtain accurate results with small computational effort and be used to solve more complicated problems.

The variations of the second frequency ($\bar{\omega}_2 = \omega_2 a^2\sqrt{\rho h/D}$) with control voltages for shells with different heights are shown in Figure 2. There are eight curves corresponding to $\gamma = 0$, 0·5, 1, 1·5, 2, 2·5, 3, 3·5 for curves from the lowest to the top at $\phi^* = 0$. Again, due to snapping, no convergent results are obtained for the non-linear static equations if the control voltages are gradually increased. Thus, curves with symbols corresponding to $\gamma = 1{\cdot}5$, 2·5, 3 are not completed in the entire range in Figure 2. From both figures, it can be seen that the frequency may increase or decrease with the control voltage depending on the shell's height and the applied voltage for the cases considered.

5. CONCLUSIONS

The differential quadrature element method is successfully used for the first time to analyze the free vibration of piezoelectric circular spherical shallow shells under control voltages. A geometrical non-linear effect is considered. Detailed formulations and solution procedures are given and numerical studies are performed. It can be seen that the DQEM results compare well with the existing data in the open literature.

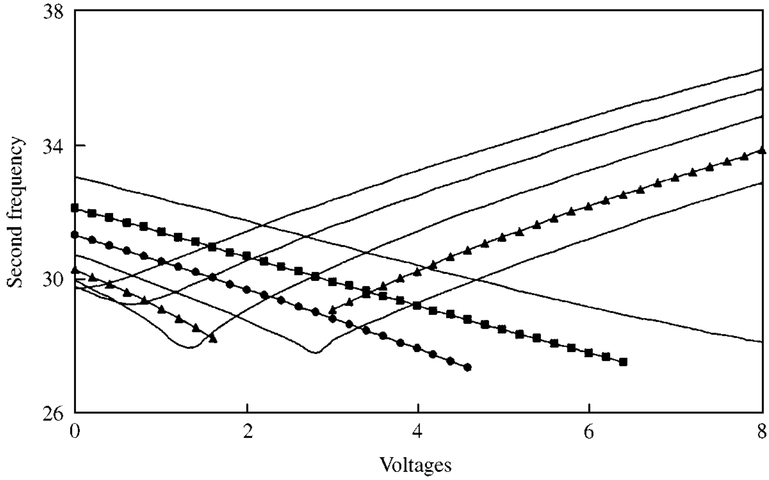

Figure 2. Variations of the second frequency ($\bar{\omega}_2 = \omega_2 a^2\sqrt{\rho h/D}$) with control voltages ϕ^* (equation (11)).

Based on the results reported herein, one may conclude that the DQEM is a useful and efficient tool for studying the dynamic behaviors of piezoelectric circular spherical shallow shells under control voltages. Accurate results are obtained by the DQEM with small computational efforts. The control voltages may increase or decrease the shell's frequency. But whether increase or decrease occurs depends on the shell's initial configurations as well as the magnitude of the control voltages.

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